

Symmetric Tamm-Dancoff q -oscillator: representation, quasi-Fibonacci nature, accidental degeneracy and coherent states

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Abstract. In this paper we propose a symmetric q -deformed Tamm-Dancoff (S-TD) oscillator algebra and study its representation, coordinate realization, and main properties. In particular, the non-Fibonacci (more exactly, quasi-Fibonacci) nature of S-TD oscillator is established, the possibility of relating it to certain p, q -deformed oscillator family shown, the occurrence of the pairwise accidental degeneracy proven. We also find the coherent state for the S-TD oscillator and show that it satisfies completeness relation. Main advantage of the S-TD model over usual Tamm-Dancoff oscillator is that due to $(q \leftrightarrow q^{-1})$ -symmetry it admits not only real, but also complex (phase-like) values of the deformation parameter q .

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1. Introduction

Complex systems in diverse fields of quantum physics, especially the systems with essential nonlinearities, can be efficiently described by quantum algebras and deformed oscillator algebras. In models of quantum field theory or particle theory, the involved Bose-like deformed oscillators (implying unusual statistics of particles and requiring modified commutation relations) can provide better description of real quantum processes, in particular with participation of pions, see [1], [2]. The use of bosonic deformed oscillators for treating phonons can significantly improve [3] matching between theory and experiment concerning the unstable phonon spectrum in ^4He . There exist interesting application of deformed bosons picture for an effective description of the spectra of excitons [4, 5]. Some version of the q -deformed Bose gas model was applied to the systems of interacting pointlike (structureless) particles in [6]. This result was recently extended in [7] where it was shown that two factors of non-ideality or deviation from the ideal Bose gas picture: (i) compositeness (two-fermionic or two-bosonic) of non-interacting Bose-like particles, and (ii) the inter-particle interactions, can be jointly accounted for by using definite model of deformed bosons. This extends the results of [8] concerning effective description by q -bosons of the interacting (thus non-ideal) gas of bosons.

Among best known deformed oscillator models, there are such ones as the Arik–Coon (AC) [9], Biedenharn–Macfarlane (BM) [10, 11], and the p, q -deformed oscillator [12] models. Their popularity explains a diversity of existing applications of these models. Much less studied is so-called Tamm–Dancoff (TD) q -deformed oscillator model [13, 14, 15]. Note nevertheless that the TD-type q -deformed Bose gas was studied in [16, 17]. Whereas AC model shows no energy level degeneracy, the TD model possesses the accidental pairwise degeneracy of levels. From the three models, only BM oscillator admits not only real, but also complex-valued deformation parameters.

In this paper, we introduce and study in some detail certain extension of the TD model, namely the ($q \leftrightarrow q^{-1}$) symmetric Tamm–Dancoff (S-TD) oscillator whose main distinction (and advantage) from the TD model is the admittance of both real and complex-valued deformation parameter q . Concerning accidental degeneracy, in the extended S-TD case situation becomes different: there is no energy level degeneracy for real q , but there appears an accidental degeneracy when q takes definite complex, phase-like, values of q .

The plan of the paper is the following. After Introduction, in Sec. 1 we present the setup for the S-TD oscillator, its representation, and the related S-TD (non-standard) version of q -calculus: q -derivative, q -integral and the S-TD q -deformed exponent. In Sec. 2, we prove the non-Fibonacci, more concretely quasi-Fibonacci properties of the S-TD q -oscillator. The next Sec. 3 is devoted to clarifying the issue of how to “embed” the S-TD q -deformed model into certain modification of the two-parameter p, q -family of deformed oscillator. The appearance of accidental pairwise energy level degeneracy is analysed in Sec. 4. In the last Sec. 5, the S-TD type deformed coherent states are

described, and their completeness is proven. The paper ends with Conclusions.

2. S-TD q -oscillator and the related version of q -calculus

To begin, we define the symmetric q -deformed bosonic Tamm-Dancoff oscillator algebra:

$$\begin{aligned} aa^\dagger - a^\dagger a &= \frac{1}{2}(1 + (1 - q^{-1})N)q^N + \frac{1}{2}(1 + (1 - q)N)q^{-N}, \\ [N, a^\dagger] &= a^\dagger, \quad [N, a] = -a, \end{aligned} \quad (1)$$

where

$$a^\dagger a = \{N\} = \frac{N}{2}(q^{N-1} + q^{-N+1}). \quad (2)$$

We can easily find the Fock-type representation of algebra (1):

$$\begin{aligned} N|n\rangle &= n|n\rangle, \quad n = 0, 1, 2, \dots, \\ a|n\rangle &= \sqrt{\frac{n}{2}(q^{n-1} + q^{-n+1})}|n-1\rangle = \sqrt{\{n\}}|n-1\rangle, \\ a^\dagger|n\rangle &= \sqrt{\frac{n+1}{2}(q^n + q^{-n})}|n+1\rangle = \sqrt{\{n+1\}}|n+1\rangle \end{aligned}$$

where $\{n\}$ denotes the S-TD type q -number defined as

$$\{n\} \equiv \frac{n}{2}(q^{n-1} + q^{-n+1}) = \frac{n}{2}q^{-n+1}(1 + (q^2)^{n-1}). \quad (3)$$

Remark that this S-TD q -bracket can be also written as a “multiplicative hybrid” of (the structure function $\varphi(n) = n$ of) usual oscillator and of the BM oscillator with the $(q \leftrightarrow q^{-1})$ -symmetric structure function $\varphi_{BM}(n) \equiv [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, namely

$$\{n\} = n \frac{[n]_q - [n-2]_q}{2} \quad \text{or} \quad \{n\} = n \frac{[2(n-1)]_q}{2[n-1]_q}. \quad (4)$$

S-TD variant of q -calculus

In order to have a functional realization of this representation, we consider the space \mathcal{P} of all monomials in variable x , and introduce its basis of monomials

$$|n\rangle \leftrightarrow \frac{x^n}{\sqrt{\{n\}!}} \quad (5)$$

where

$$\{n\}! = \prod_{k=1}^n \{k\}, \quad \{0\}! = 1.$$

Then the functional realization of the algebra (1)-(2) is given by

$$a \leftrightarrow D_x, \quad a^\dagger \leftrightarrow x, \quad N \leftrightarrow x\partial_x.$$

Here the new deformed derivative (different from well-known Jackson q -derivative [18]) is defined as

$$D_x \equiv \frac{1}{2}(T_q + T_{q^{-1}})\partial_x = \frac{1}{2}(q^{x\partial_x} + q^{-x\partial_x})\partial_x, \quad (6)$$

where $T_q = q^{x\partial_x}$ and such that $T_q f(x) = f(qx)$. It is clear that $T_{q^{-1}} = T_q^{-1}$.

The Leibnitz rule of the deformed derivative is then given by

$$D_x(f(x)g(x)) = (D_x f)(T_q^{-1}g) + (T_q f)(D_x g) - \frac{1}{2}(\mathcal{T}f)(T_q^{-1}\partial_x g) + \frac{1}{2}(T_q \partial_x f)(\mathcal{T}g),$$

where

$$\mathcal{T}f(x) = (T_q - T_q^{-1})f(x) = f(qx) - f(q^{-1}x).$$

The deformed integral is defined as

$$\begin{aligned} \int D_x f(x) &= 2 \int (T_q + T_q^{-1})^{-1} f(x) dx = \\ &= 2 \sum_{n=0}^{\infty} (-1)^n \int dx f(q^{2n+1}x) = \\ &= 2 \sum_{n=0}^{\infty} (-1)^n \int dx f(q^{-(2n+1)}x) = \\ &= \sum_{n=0}^{\infty} (-1)^n \int dx (f(q^{2n+1}x) + f(q^{-(2n+1)}x)). \end{aligned} \quad (7)$$

The last equality in (7), showing the explicit $(q \leftrightarrow q^{-1})$ -symmetry, follows due to identity

$$2(T_q + T_{q^{-1}})^{-1} = T_{q^{-1}}(1 + T_{q^{-2}})^{-1} + T_q(1 + T_{q^2})^{-1}.$$

Applying S-TD derivative and S-TD integral to x^n yields

$$\begin{aligned} D_x x^n &= \{n\} x^{n-1}, \quad \int D_x x^n = \frac{x^{n+1}}{\{n+1\}}, \\ \int D_x \frac{1}{x} &= \frac{2}{q + q^{-1}} \ln x, \quad D_x \left(\frac{2}{q + q^{-1}} \ln x \right) = \frac{1}{x}, \end{aligned} \quad (8)$$

and also

$$D_x^k x^n = \frac{\{n\}!}{\{n-k\}!} x^{n-k}.$$

The S-TD-exponential function $\mathcal{E}(x)$ is defined as[‡]

$$\mathcal{E}(x) = \sum_{n=0}^{\infty} \frac{1}{\{n\}!} x^n, \quad (9)$$

or

$$\mathcal{E}(x) = 1 + \sum_{n=1}^{\infty} \frac{2^n q^{\frac{1}{2}n(n-1)}}{n! \prod_{k=0}^{n-1} (1 + (q^2)^k)} x^n. \quad (10)$$

It is worth to note that, whereas the TD-type q -exponent for which the convergence of the series analogous to (9), with respect to ordinary $\exp(x)$, improves at $q > 1$ and worsens at $q < 1$, the convergence of the series in (9) - (10) is better for any positive q .

[‡] Note that, unlike well-known q -exponents $e_q(x)$ and $E_q(x)$ (given e.g. in [19]), the $\mathcal{E}(x)$ possesses straightforward $q \rightarrow 1$ limit to the usual exponent.

One can verify that the inverse $[\mathcal{E}(x)]^{-1}$ differs from $\mathcal{E}(-x)$. Let us give the $\mathcal{E}^{-1}(x)$ explicitly, namely

$$\frac{1}{\mathcal{E}(x)} = \sum_{n=0}^{\infty} b_n x^n \quad (11)$$

where $b_0 = 1$, and all the other expansion coefficients are found using recursion relation

$$b_n = -\frac{1}{\{n\}} - \sum_{j=1}^{n-1} b_j \frac{1}{\{n-j\}!}, \quad n \geq 1,$$

inferred from the equality: $\frac{1}{\mathcal{E}(x)} \sum_{n=0}^{\infty} \frac{x^n}{\{n\}!} = 1$. Let us write first few coefficients:

$$b_0 = 1, \quad b_1 = -1, \quad b_2 = -\frac{1}{\{2\}!} + 1, \quad b_3 = -\frac{1}{\{3\}!} + \frac{2}{\{2\}!} - 1,$$

$$b_4 = -\frac{1}{\{4\}!} + \frac{2}{\{3\}!} - \frac{3}{\{2\}!} + \frac{1}{(\{2\}!)^2} + 1,$$

$$b_5 = -\frac{1}{\{5\}!} + \frac{2}{\{4\}!} - \frac{3}{\{3\}!} + \frac{4}{\{2\}!} + \frac{2}{\{3\}!\{2\}!} - \frac{3}{(\{2\}!)^2} - 1.$$

The S-TD q -exponent $\mathcal{E}(x)$ can be also presented in yet another form. Namely,

$$\mathcal{E}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{n!(-1, q^2)_n} (2x)^n \quad (12)$$

where the notation $(a, Q)_n$ means:

$$(a, Q)_n \equiv (1-a)(1-aQ)(1-aQ^2) \cdots (1-aQ^{n-1}), \quad (a, Q)_0 = 1.$$

It is easy to check that the S-TD-exponential function satisfies

$$D_x \mathcal{E}(ax) = a \mathcal{E}(ax), \quad \int D_x \mathcal{E}(ax) = \frac{1}{a} \mathcal{E}(ax) \quad (13)$$

for an arbitrary constant a . From the relation (assume $a > 0$)

$$\int_0^{\infty} D_x \mathcal{E}(-ax) = \frac{1}{a}, \quad (14)$$

we can obtain the following formula

$$\int_0^{\infty} D_x \mathcal{E}(-ax) x^n = \frac{(-1)^n}{a^{n+1}} \prod_{k=1}^n \{-k\}. \quad (15)$$

Inserting $a = 1$ into eq.(15) yields

$$\int_0^{\infty} D_x \mathcal{E}(-x) x^n = (-1)^n \prod_{k=1}^n \{-k\}. \quad (16)$$

The latter relation can be rewritten in the form

$$\int_0^{\infty} D_x \mathcal{E}(-x) x^n = \frac{2\{n+2\}!}{\{2\}(n+1)(n+2)} \quad (17)$$

if one uses the formula

$$\{-k\} = -\frac{k}{k+2} \{k+2\}.$$

As last point in this section, let us show that the S-TD exponential given in (11) can be presented as some bibasic hypergeometric function [19] whose general definition is

$$\begin{aligned} {}_{r,r'}\mathcal{F}_{s,s'}(\underline{a}_r; \underline{c}_{r'} | \underline{b}_s; \underline{d}_{s'}; (q, p); z) &\equiv \\ &\equiv \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n (c_1, c_2, \dots, c_{r'}; p)_n}{(b_1, b_2, \dots, b_s; q)_n (d_1, d_2, \dots, d_{s'}; p)_n (q; q)_n} \times \\ &\times \left[(-1)^n q^{n(n-1)/2} \right]^{1+s-r} \left[(-1)^n p^{n(n-1)/2} \right]^{s'-r'} z^n. \end{aligned}$$

Here $\underline{a}_m = (a_1, a_2, \dots, a_m)$ and $(a_1, a_2, \dots, a_m; q)_n \equiv \prod_{k=1}^m (a_k; q)_n$; likewise, $(c_1, c_2, \dots, c_{m'}; p)_n \equiv \prod_{l=1}^{m'} (c_l; p)_n$. Indeed, taking into account that

$$(a^2, q^2)_n = (a, q)_n (-a, q)_n, \quad \lim_{p \rightarrow 1} \frac{(p, p)_n}{(1-p)^n} = n!$$

from (11) we obtain the desired result:

$$\mathcal{E}(x) \equiv \exp_q^{(\text{S-TD})}(z) \equiv \lim_{p \rightarrow 1} {}_{0,1}\mathcal{F}_{0,2}(-; 0 | -; i, -i; (p, q); (1-p)z). \quad (18)$$

Thus, like in the case on non-symmetric TD q -exponential

$$\exp_q^{(\text{TD})} = \sum_{n=0}^{\infty} \frac{q^{-\frac{1}{2}n(n-1)}}{n!} x^n$$

considered in [23], the symmetric S-TD type q -exponential $\mathcal{E}(x)$ studied in this paper also takes the form of some (limit of) bibasic hypergeometric function.

3. Non-Fibonacci (Quasi-Fibonacci) nature of S-TD oscillator

Diverse deformed oscillator models appeared in quantum physics till present time [9, 10, 11, 12, 13, 14, 15], [23, 21, 22]. All of them have different structure functions and different properties. Since some of them were known as Fibonacci ones [21], the question arose about some classification of deformed oscillator models. As a criterion for possible classification of DOs, in [22] we proposed to check the validity (or failure) of the Fibonacci-like recurrence relation

$$E_{n+1} = \lambda E_n + \rho E_{n-1}, \quad n \geq 1, \quad E_n = \frac{1}{2}(\phi_{n+1} + \phi_n), \quad (19)$$

for each three consecutive energy levels, with *constant* real λ and ρ . Note that usual quantum harmonic oscillator is the Fibonacci one: it satisfies eq. (19) with $\lambda = 2$ and $\rho = -1$.

In this section we will check whether symmetric Tamm-Dancoff q -deformed oscillator defined in (1) belongs to the class of Fibonacci oscillators. To do that, we substitute the corresponding expressions for the energy of S-TD deformed oscillator in (19):

$$\begin{aligned} (n+1)(q^n + q^{-n}) + (n+2)(q^{n+1} + q^{-n-1}) &= \\ &= \lambda(n(q^{n-1} + q^{-n+1}) + (n+1)(q^n + q^{-n})) + \end{aligned}$$

$$+ \rho \left((n-1)(q^{n-2} + q^{-n+2}) + n(q^{n-1} + q^{-n+1}) \right).$$

From this, considering independent basis elements, we infer the following set of equations:

$$\begin{aligned} nq^n : \quad & 1 + q = \lambda(q^{-1} + 1) + \rho(q^{-2} + q^{-1}); \\ q^n : \quad & 1 + 2q = \lambda - \rho q^{-2}; \\ nq^{-n} : \quad & 1 + q^{-1} = (q + 1)(\lambda + \rho q); \\ q^{-n} : \quad & 1 + 2q^{-1} = \lambda - \rho q^2. \end{aligned} \tag{20}$$

The first two equations are solved to give $\lambda = 2q$ and $\rho = -q^2$. These values of λ and ρ , however, cannot satisfy third and fourth equations if $q \neq 1$. Therefore we conclude that the system of equations (20) is incompatible and the symmetric Tamm-Dancoff oscillator does not belong to the class of Fibonacci oscillators.

Following the ideas of [22], we consider for the S-TD oscillator the *quasi-Fibonacci* recurrence relation that involves the coefficients $\lambda = \lambda(n) \equiv \lambda_n$ and $\rho = \rho(n) \equiv \rho_n$:

$$E_{n+1} = \lambda_n E_n + \rho_n E_{n-1}, \quad n \geq 1. \tag{21}$$

To certify the quasi-Fibonacci nature of the S-TD oscillator we have to find λ_n and ρ_n explicitly. There exist different ways to find λ_n and ρ_n , and here we follow two of them.

First way ("simplest splitting"). Replace the quasi-Fibonacci relation (21) by the system of two equations for λ_n and ρ_n in terms of the structure function $\varphi(n)$, see (26):

$$\begin{cases} \varphi(n+1) = \lambda_n \varphi(n) + \rho_n \varphi(n-1), \\ \varphi(n+2) = \lambda_n \varphi(n+1) + \rho_n \varphi(n). \end{cases} \tag{22}$$

From these we find

$$\lambda_n = \frac{\varphi(n+1) - \rho_n \varphi(n-1)}{\varphi(n)}, \quad \rho_n = \frac{\varphi(n+2)\varphi(n) - \varphi^2(n+1)}{\varphi^2(n) - \varphi(n+1)\varphi(n-1)}. \tag{23}$$

Recalling explicit structure function $\varphi(n)$ of S-TD deformed oscillator, we finally obtain

$$\lambda_n = \frac{2(1 + q^{-2n})(q + q^{2n-1}) - (n-1)(n+2)(q + q^{-1})(q - q^{-1})^2}{(1 + q^{-2n})(q^2 + q^{2n-2}) - n^2(q - q^{-1})^2}, \tag{24}$$

$$\rho_n = -\frac{(q^n + q^{-n})^2 - n(n+2)(q + q^{-1})^2}{(1 + q^{-2n})(q^2 + q^{2n-2}) - n^2(q - q^{-1})^2}. \tag{25}$$

Note that in the limit $q \rightarrow 1$ we have $\lambda_n \rightarrow 2$ and $\rho_n \rightarrow -1$ as for the usual quantum oscillator. It is easy to verify that since the coefficients λ_n and ρ_n in (23) – (25) are the solutions of the system of equations (22), they obviously satisfy the quasi-Fibonacci relation (21).

Following second, most general, way (see [22]) for deriving the coefficients λ_n and ρ_n , we consider the system

$$\begin{cases} E_{n+1} = \lambda_n E_n + \rho_n E_{n-1}, \\ E_{n+2} = \lambda_{n+1} E_{n+1} + \rho_{n+1} E_n, \end{cases} \tag{26}$$

which cannot be solved uniquely (high degree of arbitrariness). Therefore we take the first relation in (26) and use the explicit structure function of S-TD oscillator:

$$\begin{aligned} & (n+1)(q^n + q^{-n}) + (n+2)(q^{n+1} + q^{-n-1}) = \\ & = \lambda_n(n(q^{n-1} + q^{-n+1}) + (n+1)(q^n + q^{-n})) + \\ & + \rho_n((n-1)(q^{n-2} + q^{-n+2}) + n(q^{n-1} + q^{-n+1})). \end{aligned} \quad (27)$$

The latter equation can be replaced equivalently by the two equations:

$$0 = (\lambda_n - 2)(n+1)(q^n + q^{-n}) - K(q, n)(n+1)(q^n + q^{-n}), \quad (28)$$

$$\begin{aligned} & (n+2)(q^{n+1} + q^{-n-1}) - (n+1)(q^n + q^{-n}) = (\lambda_n + \rho_n)n(q^{n-1} + q^{-n+1}) + \\ & + \rho_n(n-1)(q^{n-2} + q^{-n+2}) + K(q, n)(n+1)(q^n + q^{-n}), \end{aligned} \quad (29)$$

where $K(q, n)$ is an arbitrary function incorporating the arbitrariness of splitting eq. (27) into two equations, and such that $K(q, n) \rightarrow 0$ when $q \rightarrow 1$. Now, from the equation (28) one easily finds λ :

$$\lambda_n = 2 + K(q, n), K(q, n) \xrightarrow{q \rightarrow 1} 0. \quad (30)$$

Substituting this λ_n in (28), we obtain the coefficient ρ_n , namely

$$\begin{aligned} \rho_n = & \frac{(n+2)(q^{n+1} + q^{-n-1}) - (n+1)(q^n + q^{-n}) - 2n(q^{n-1} + q^{-n+1})}{n(q^{n-1} + q^{-n+1}) + (n-1)(q^{n-2} + q^{-n+2})} - \\ & - \frac{K(q, n)(n(q^{n-1} + q^{-n+1}) + (n+1)(q^n + q^{-n}))}{n(q^{n-1} + q^{-n+1}) + (n-1)(q^{n-2} + q^{-n+2})}. \end{aligned} \quad (31)$$

Note that at $q \rightarrow 1$ we obtain $\lambda_n \rightarrow 2$, $\rho_n \rightarrow -1$ as it should be (the usual oscillator case).

It is useful to check two special cases of $K(q, n)$:

1. If $K(q, n) = 0$, we have

$\lambda_n = 2$, and

$$\rho_n = \frac{(n+2)(q^{n+1} + q^{-n-1}) - (n+1)(q^n + q^{-n}) - 2n(q^{n-1} + q^{-n+1})}{n(q^{n-1} + q^{-n+1}) + (n-1)(q^{n-2} + q^{-n+2})} \xrightarrow{q \rightarrow 1} -1.$$

2. On the other hand, it is possible that

$$\rho_n = -1 \quad \text{if} \quad K(q, n) = -1 + \widetilde{K}(q, n), \quad \text{where}$$

$$\widetilde{K}(q, n) = \frac{(n+2)(q^{n+1} + q^{-n-1}) + (n-1)(q^{n-2} + q^{-n+2})}{(n+1)(q^n + q^{-n}) + n(q^{n-1} + q^{-n+1})}.$$

Then, $\lambda_n = 1 + \widetilde{K}(q, n)$. At $q \rightarrow 1$, we have $\widetilde{K}(q, n) \rightarrow 1$, $\lambda_n \rightarrow 2$. Remark that both in case 1 and case 2, one coefficient is still n -dependent.

4. Relating S-TD oscillator with p, q -deformed oscillators

It is an interesting question whether the one-parameter symmetric Tamm-Dancoff can be “embedded” in the two-parameter deformed oscillator. Consider the five-parameter oscillator whose structure function is build by “gluing” (using the weight-like t -parameter, see formula (22) in [23]) the two different structure functions (here – two different copies of p, q -oscillator):

$$\begin{aligned}\tilde{\varphi}_{p,q,P,Q;t} &= t\varphi_{p,q} + (1-t)\varphi_{P,Q}, \quad \text{where} \\ \varphi_{p,q} &= \frac{p^N - q^N}{p - q}, \quad \varphi_{P,Q} = \frac{P^N - Q^N}{P - Q}.\end{aligned}\tag{32}$$

Recall that p, q -deformed oscillator with structure function $\varphi_{p,q}$ is the Fibonacci one [21]. The same is true for the P, Q -deformed oscillator defined by $\varphi_{P,Q}$. But, that is not true in general for the hybrid deformed oscillator given by $\tilde{\varphi}_{p,q,P,Q;t}$ (see the Proposition below).

Let us put $P = p^{-1}$, $Q = q^{-1}$, and $t = \frac{1}{2}$. That yields

$$\tilde{\varphi}_{p,q} = \frac{1}{2}(\varphi_{p,q} + \varphi_{p^{-1},q^{-1}}),\tag{33}$$

and it is evident that in the special case of $p = q$, this two-parameter deformed oscillator yields nothing but the (structure function of) S-TD deformed oscillator.

Proposition. The five-parameter $(t; p, q, P, Q)$ -deformed oscillator given by the structure function $\tilde{\varphi}_{p,q,P,Q;t}$ in (32) is not Fibonacci (except for the two cases: $p = P$, $q = Q$ and $P = q^{-1}$, $Q = p^{-1}$). Thus, it should be viewed as quasi-Fibonacci one.

The proof proceeds similarly to that based on (20), but now the analogous system of 4 equations stems from equating the coefficient expressions (involving $q, p, Q, P, \lambda, \rho$) at p^n, q^n, P^n and Q^n viewed as independent basis elements.

Remark. An alternative viewpoint can be also adopted: deformed oscillator with the structure function (1) can be viewed as two-dimensional or two-mode deformed oscillator whose two modes are (i) independent, and (ii) each one is described by a copy of mutually dual (in the sense of $q \rightarrow q^{-1}$, $p \rightarrow p^{-1}$ duality) p, q -oscillators and likewise, at $p = q$, by a copy of two mutually ($q \leftrightarrow q^{-1}$)-dual TD oscillators.

5. Pairwise energy levels degeneracy of S-TD oscillator

Taking the Hamiltonian as

$$H = \frac{1}{2}(a^\dagger a + a a^\dagger),$$

we have its eigenvalues

$$H|n\rangle = E_n|n\rangle$$

where

$$E_n = \frac{1}{4}[(n+1)(q^n + q^{-n}) + n(q^{n-1} + q^{1-n})]\tag{34}$$

(note that $E_0 = \frac{1}{2}$, as for the usual non-deformed quantum oscillator).

To study possible accidental degeneracy of, say, neighboring levels, let us consider

$$\begin{aligned}\Delta E_n &= E_{n+1} - E_n = \\ &= \frac{1}{4}[(n+2)(q^{n+1} + q^{-n-1}) - n(q^{n-1} + q^{-n+1})] = \\ &= \frac{1}{4}[2(q^{n+1} + q^{-n-1}) + n(q - q^{-1})(q^n - q^{-n})] > 0.\end{aligned}\tag{35}$$

The latter follows for $q > 0$ due to the inequality $q + q^{-1} \geq 2$ (saturated at $q = 1$), and we conclude that there is no degeneracy for real positive values of q .

The situation however differs if q is phase-like. So, let us assume

$$q = \exp(i\theta), \quad -\pi \leq \theta \leq \pi.$$

In this case, from (35) we have:

$$\begin{aligned}E_{n+1} - E_n &= \frac{1}{2}[(n+2)\cos((n+1)\theta) - n\cos((n-1)\theta)] = \\ &= \cos(n\theta)\cos\theta - (n+1)\sin(n\theta)\sin\theta = \\ &= \frac{1}{2}[(n+2)T_{n+1}(x) - nT_{n-1}(x)] = \\ &= (n+1)T_{n+1}(x) - nxT_n(x), \quad x \equiv \cos\theta,\end{aligned}$$

where $T_n(\cos\theta)$ is the Chebyshev polynomial of n -th order.

It is possible to solve $E_{n+1} - E_n = 0$ at any fixed n . As instances, consider few cases: $n = 0, 1, 2, 3$.

$$\underline{E_1 = E_0}. \quad \text{This holds at } \theta = \theta_0 = \pm\frac{\pi}{2}, \quad \text{or } q = \pm i.$$

$$\underline{E_2 = E_1}. \quad \text{This holds at } \theta = \theta_1 = \pm \arcsin \frac{\sqrt{3}}{3}.$$

$$\underline{E_3 = E_2}. \quad \text{This holds at } \theta = \theta_2 = \pm \arcsin \frac{\sqrt{2}}{4} \text{ (and also } \theta = \theta_0).$$

$$\underline{E_4 = E_3}. \quad \text{This holds at } \theta = \theta_3 = \pm \arcsin \sqrt{\frac{17 \pm \sqrt{209}}{40}}.$$

In a similar fashion one can prove that other cases of pairwise degeneracy may occur. Namely, there exist special value(s) of the parameter $q = \exp(i\theta)$ that solve the equation

$$E_{n+r} = E_n, \quad n \geq 0, \quad r > 1.$$

The latter can be presented as

$$\begin{aligned}&(n+r+1)(q^{n+r} + q^{-n-r}) + (n+r)(q^{n+r-1} + q^{-n-r+1}) - \\ &-(n+1)(q^n + q^{-n}) - n(q^{n-1} + q^{-n+1}) = 0\end{aligned}$$

or, in terms of Chebyshev polynomials, as

$$(n+r+1)T_{n+r}(x) + (n+r)T_{n+r-1}(x) - (n+1)T_n(x) - nT_{n-1}(x) = 0, \quad x \equiv \cos \theta,$$

or as

$$(n+1)(T_{n+r}(x) - T_n(x)) + n(T_{n+r-1}(x) - T_{n-1}(x)) + r(T_{n+r}(x) + T_{n+r-1}(x)) = 0.$$

For instance of solving, we quote two cases:

$$\underline{E_2 = E_0}. \quad \text{This holds at } \theta = -\pi \quad \text{or at } \theta = \arccos \frac{2}{3}.$$

$$\underline{E_3 = E_1}. \quad \text{This holds at } \theta = -\pi \quad \text{or at } \theta = \arccos \frac{5 \pm \sqrt{89}}{16}.$$

6. Coherent states

In this section, we construct the coherent states of the STD-oscillator algebra (1)-(2). The coherent state $|z\rangle$ is defined as an eigenstate of the annihilation operator in the form

$$a|z\rangle = z|z\rangle. \quad (36)$$

The coherent state can be also represented by using the Fock eigenvectors of the number operator:

$$|z\rangle = \sum_{n=0}^{\infty} c_n(z)|n\rangle. \quad (37)$$

Inserting eq.(37) into eq.(36), we have

$$\sum_{n=1}^{\infty} c_n(z)\sqrt{\{n\}}|n-1\rangle = \sum_{n=0}^{\infty} z c_n(z)|n\rangle. \quad (38)$$

From eq.(38), we get the following recurrence relation

$$c_{n+1} = \frac{z}{\sqrt{\{n+1\}}} c_n \quad (n = 0, 1, 2, \dots). \quad (39)$$

Solving this relation (39), we infer

$$c_n = \frac{1}{\sqrt{\{n\}}!} z^n c_0.$$

From $\langle z|z\rangle = 1$ it is not difficult to obtain

$$c_0^{-2} = \mathcal{E}(|z|^2). \quad (40)$$

Thus the coherent state takes the form

$$|z\rangle = \frac{1}{\sqrt{\mathcal{E}(|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\{n\}}!} |n\rangle. \quad (41)$$

Now let us show that the coherent state $|z\rangle$ forms a complete set of states. To establish this, we invoke the completeness relation:

$$\frac{1}{\pi} \int \int |z\rangle \mu(|z|^2) \langle z| |z| D|z| d\theta = I, \quad (42)$$

where $\mu(|z|^2)$ is a weight function. Inserting eq.(41) in eq.(42), we obtain

$$\sum_{n=0}^{\infty} \frac{1}{\{n\}!} |n\rangle \langle n| \int_0^{\infty} f(x) x^n Dx = I \quad (43)$$

where $x = |z|^2$ and $f(x) = \mu(x)/\mathcal{E}(x)$. If we find $f(x)$ satisfying

$$\int_0^{\infty} f(x) x^n Dx = \{n\}! , \quad (44)$$

the completeness is proved. It is useful to set

$$f(x) = \mathcal{E}(-x)g(x) = \mathcal{E}(-x) \sum_{k=0}^{\infty} g_k x^k, \quad (45)$$

that implies $\mu(x) = \mathcal{E}(x)\mathcal{E}(-x)g(x)$.

Inserting eq.(45) into eq.(43), with the account of (17) we obtain

$$\sum_{k=0}^{\infty} g_k \frac{(n+k)!}{n!} \prod_{j=n+1}^{n+k+2} \phi(j) = \phi(2), \quad \phi(j) \equiv \frac{\{j\}}{j}. \quad (46)$$

From (46) we infer the recurrence relation:

$$g_k = \frac{(-1)^k}{k!\phi(k+2)!} \left[\phi(2) - \sum_{i=0}^{k-1} g_i \frac{(-1)^i k!}{(k-i)!} \prod_{j=k-i+1}^{k+2} \phi(j) \right], \quad (k \geq 1)$$

$$g_0 = 1, \quad \phi(i)! = \prod_{j=1}^i \phi(j)$$

One easily gets the particular coefficients g_k :

$$g_0 = 1, \quad g_1 = \frac{\phi(2)(\phi(3) - 1)}{\phi(3)!},$$

$$g_2 = \frac{\phi(2) - \phi(3)\phi(4) + 2\phi(2)(\phi(3) - 1)\phi(4)}{2!\phi(4)!},$$

and so on. This shows in a constructive way the existence of the function $g(x)$ and, through $\mu(x) = \mathcal{E}(x)\mathcal{E}(-x)g(x)$, of the weight function $\mu(|z|^2)$. Thus, the completeness is proved.

7. Conclusions

Within the symmetric q -deformed Tamm-Dancoff oscillator proposed in this paper and possessing the symmetry $q \leftrightarrow q^{-1}$ we explored a number of its aspects. We constructed both the Fock type representation, and the coordinate space realization. For the needs of the latter we introduced and studied in some detail the corresponding non-standard version of q -calculus, first of all the S-TD type q -differentiation and q -integration. The related S-TD type q -exponential function given in (9) or (11), possesses interesting properties (e.g. under q -integration), as shown in (11) and (13) – (17). Besides, we have found its inverse and its presentation, see (18), as a particular case of bibasic hypergeometric function.

Among the special properties possessed by S-TD q -oscillator, we establish its quasi-Fibonacci (thus non-Fibonacci) nature and the ability to present the S-TD q -oscillator as a special case of some family of p, q -deformed oscillators. Let us emphasize that the result concerning quasi-Fibonacci nature of S-TD q -oscillator (see Sec. 3 and especially Sec. 4) points the simple way how to build many new quasi-Fibonacci oscillators taking two (or more) Fibonacci ones as building blocks. Because of $q \leftrightarrow q^{-1}$ -symmetry, the studied q -oscillator admits not only real, but also phase-like deformation parameter and, due to this fact, the possibility of pairwise accidental energy level degeneracy arises. At last, we provide the S-TD type q -coherent states and prove for these the completeness relation.

It would be interesting to study, first, the connection of the S-TD q -oscillator with respective deformation of Heisenberg algebra for the position and momentum operators like in [24] and, second, the thermostatics of S-TD type q -deformed Bose gas model (linked with the S-TD q -oscillators). Also, we hope the proposed symmetric q -deformed Tamm-Dancoff oscillator will find interesting physical applications.

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